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1999 J. Phys. A: Math. Gen. 32 3805

(http://iopscience.iop.org/0305-4470/32/20/312)

J. Phys. A: Math. Gen. **32** (1999) 3805–3814. Printed in the UK PII: S0305-4470(99)00019-0

Structures of electromagnetic type on vector bundles

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Received 3 December 1998, in final form 17 March 1999

Abstract. Structures of electromagnetic type on a vector bundle are introduced and studied. The metric case is also defined and studied. The sets of compatible connections are determined and a canonical connection is defined.

1. Introduction

Structures of electromagnetic type (em-structures) and structures of metric electromagnetic type (mem-structures) on a manifold were progressively introduced in [7, 9, 11] (see also [6]) and studied in detail in [5, 7, 8, 13, 14]. In the present paper we define similar structures for the case of a vector bundle $\xi = (E, \pi, M)$, and relate them to product, complex, para-Hermitian, Hermitian, para-Kähler or indefinite Kähler, structures. (In the following, by a pseudo-Riemannian metric we shall understand a metric of any signature, and by an indefinite (metric) structure a structure including a pseudo-Riemannian metric.) Then, we determine the set of connections on ξ compatible with those structures and we introduce a canonical connection. Considering an almost-para-Hermitian (respectively, indefinite Hermitian) structure on the base manifold M and an indefinite Hermitian (respectively, para-Hermitian) structure of the bundle ξ, we prove that the corresponding diagonal lift of these structures, with respect to a connection on ξ, are mem-structures on the total space E. Finally, some properties of those mem-structures are established.

We recall the physical origin of the topic [9, 11]. Let M^4 be a spacetime of general relativity, with gravitational tensor g of signature $-{+}{+}{+}$. Let F be the electromagnetic field of type $(0, 2)$, which is skewsymmetric, that is a 2-form. Setting $F(X, Y) = g(JX, Y)$, the tensor field J thus defined is the electromagnetic tensor field of type $(1, 1)$ associated to F. We have $g(JX, Y) + g(X, JY) = 0$. The characteristic equation of J is $\det(J - \lambda I) = 0$, which is satisfied by J, and we have

$$J^4 + 2kJ^2 + lI = 0 \qquad k = -\tfrac{1}{4}\,\mathrm{trace}\, J^2 \qquad l = \det J.$$

If $x \in M^4$, it is said that J_x is of first, second or third class at x if, respectively,

$$l_x \neq 0 \qquad l_x = 0 \quad k_x \neq 0 \qquad l_x = 0 \quad k_x = 0.$$

It is said that J is of first, second or third class if it is of such a class at every x. The characteristic polynomial of the second class is $J^2(J^2+2k)$, but the minimal polynomial is $J(J^2+2k)$, so that the condition $J(J^2+2k)=0$ characterizes the second class. The field of an electromagnetic plane wave is of third class. The field of a moving electron is of second class. More complicated fields belong to the first class. The equation one obtains from the minimal polynomial in the first class is

$$(J^2-f^2)(J^2+h^2)=0 \tag{1.1}$$

with f, h nowhere-vanishing C^∞ functions on M^4. Such a tensor field J on a general manifold M determines a G-structure on M.

To handle the nonconstant local cross section situation corresponding to (1.1), one can use the relationships among G-structures, related sections of an associated bundle and functions of a certain kind on M, as follows: let $(\mathcal{P}, \pi_P, M, H)$ be a principal bundle with group H, $H\times W \to W$ a left action of H on a manifold W, and $(E=\mathcal{P}\times_H W, \pi_E, M, W)$ the associated bundle. A J-subset S of W with corresponding group G, a subgroup of H, is defined by the conditions: (a) $S\subset$ fixpoint set of G, (b) $h\in H$, $h(S)\cap S\neq\emptyset \Rightarrow h\in G$. For instance, points are J-subsets with G the corresponding isotropy group. A cross section K of π_E is a J-section if it can be locally represented as the 'product' of a cross section σ of π_P and a S-valued function $\widetilde{K}$, so that

$$K_x=\sigma_x\cdot\widetilde{K}_x=\text{equivalence class of }(\sigma_x,\widetilde{K}_x)\text{ in }E.$$

Then $\widetilde{K}$ is globally defined, and the σ generate a principal subbundle of $\mathcal{P}$. K is a constant J-section if and only if $\widetilde{K}$ is constant. Different sections can generate the same subbundle, and in fact, every principal subbundle can be generated by a constant J-section.

Now, let $\mathcal{P}$ be the principal bundle of frames over M, so that $H=GL(n,\mathbb{R})$, and let W be a real vector space. If $J\in W$ is given with the conditions stated above, a J-section generates a J-structure with group G, which is a G-structure. The tensor K has in principle *variable components* in adapted frames. This is a slight generalization with respect to the usually considered G-structures, given by tensors with constant components, which here correspond to constant J-sections. Since every J-structure is generated by some constant J-section, this generalization is useless for the study of the J-structure itself; but if the emphasis shifts to the study of variable J-sections, the results are significant, especially with respect to the parallelizability of the tensors.

In the particular case of a (1, 1) tensor field J satisfying $(J^2-f^2)(J^2+h^2)=0$, with characteristic polynomial $(x-p)^{r_1}(x-p)^{r_2}(x^2+q^2)^s$, $r_1, r_2, s\geqslant 1$, $r_1+r_2+2s=n=\dim M$, the J-subset consists of matrices of the form

$$\begin{pmatrix} pI_{r_1} & & & \\ & -pI_{r_2} & & \\ & & & -qI_s \\ & & qI_s & \end{pmatrix}$$

and the structural group is $G=GL(r_1,\mathbb{R})\times GL(r_2,\mathbb{R})\times GL(s,\mathbb{C})$. It is proved [7] that the G-structure defined by J above is also defined by a tensor field, say again J, satisfying $(J^2-1)(J^2+1)=0$, that is, the relation $J^4=1$ considered in the present paper.

Notice that *the G-structure is exactly the same, not an associated or equivalent one*. In the four-dimensional case the group reduces to $G=GL(1,\mathbb{R})\times GL(1,\mathbb{R})\times GL(1,\mathbb{C})$. It is also proved [7] that there exists an adapted Riemannian metric so that the group can be reduced to $G=O(r_1)\times O(r_2)\times U(s)$, and in the four-dimensional case to $\mathbb{Z}_2\times\mathbb{Z}_2\times U(1)$, that is, essentially to the unitary group $U(1)$.

2. Structures of electromagnetic type on a vector bundle

Let $\xi = (E, \pi, M)$ be a C^∞ vector bundle with total space E and projection map π over a connected paracompact base manifold M. The rank of E is the (common) dimension of the fibres. Let $C^\infty(M)$ denote the ring of real functions, $\mathcal{T}_q^p(M)$ the $C^\infty(M)$-module of (p, q)-tensor fields, and $\mathcal{T}(M)$ the $C^\infty(M)$-tensor algebra of M. We, respectively, denote by $\mathcal{T}_q^p(\xi)$ and $\mathcal{T}(\xi)$ the $C^\infty(M)$-module of tensor fields of type (p, q) and the $C^\infty(M)$-tensor algebra of the bundle ξ.

We recall that an almost-product (respectively, almost-complex) structure on a manifold M is defined by a tensor field J of type (1, 1) satisfying $J^2 = I$ (respectively, $J^2 = -I$). An almost-para-Hermitian (respectively, indefinite almost-Hermitian) structure on M is defined by a pair (J, g), given by an almost-product (respectively, almost-complex) structure J and a pseudo-Riemannian metric compatible with J in the sense that $g(JX, Y) + g(X, JY) = 0$, $X, Y \in \mathfrak{X}(M)$; that is, as an anti-isometry (respectively, isometry). A para-Kähler (respectively, indefinite Kähler) manifold is a manifold M endowed with an almost-para-Hermitian (respectively, indefinite almost-Hermitian) structure such that the Levi-Civita connection of g parallelizes J.

Definition 2.1. *A structure of electromagnetic type on $\xi = (E, \pi, M)$ is an M-endomorphism J of ξ satisfying*

$$J^4 = I$$

with characteristic polynomial $(x-1)^{r_1}(x+1)^{r_2}(x^2+1)^s$, where r_1, r_2, s are constants greater than or equal to 1 *such that* $r_1 + r_2 + 2s = \text{rank } E$.

Setting $P = J^2$, we have $P^2 = I$, so P is a product structure on ξ, admitting J as a 'square root'. Conversely, if P is a product structure admitting a 'square root' J, then J is an em-structure on ξ. Denoting by ξ_1 and ξ_2, respectively, the +1 and -1 eigen-subbundles of P, it is easy to see that ξ_1 and ξ_2 are invariant by J and that $J_1 = J|_{\xi_1}$ defines a product structure of ξ_1 and $J_2 = J|_{\xi_2}$ a complex structure of ξ_2. So, one has

$$\xi = \xi_1 \oplus \xi_2 \qquad J = J_1 \oplus J_2. \tag{2.1}$$

Conversely, if ξ_1 and ξ_2 are two supplementary subbundles of ξ, J_1 is a product structure of ξ_1, and J_2 a complex structure of ξ_2, then $J = J_1 \oplus J_2$ is an em-structure on ξ. Denoting by P_1 and P_2 the projections of ξ on ξ_1 and ξ_2, respectively, we obtain

$$P = P_1 - P_2 \qquad J = J_1 \circ P_1 + J_2 \circ P_2.$$

Summing up we have

Proposition 2.1. *An em-structure on the vector bundle $\xi = (E, \pi, M)$ can be defined by each one of the following conditions:*

(a) An M-endomorphism J of ξ satisfying $J^4 = I$.
(b) A product structure P of ξ admitting a 'square root' J.
(c) Two supplementary subbundles ξ_1 and ξ_2 of ξ, respectively, endowed with a product structure and a complex structure.

Remark 2.1. *A product structure P which admits a 'square root' is a particular one because* rank ξ_2 *must be even.*

Definition 2.2. *A structure of metric electromagnetic type (mem-structure) on the vector bundle ξ is a pair (J, g), where J is an em-structure and g a pseudo-Riemannian metric on ξ satisfying the compability condition*

$$g(JX, Y) + g(X, JY) = 0 \qquad X, Y \in \xi. \tag{2.2}$$

Denoting by δ_J the derivation defined by J in the tensor algebra $\mathcal{T}(\xi)$, the relation (2.2) can be written as

$$\delta_J g = 0$$

from which it follows that $g(PX, PY) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$. Therefore, the pair (P, g) is a pseudo-Riemannian product structure of ξ and so the subbundles ξ_1 and ξ_2 are mutually orthogonal with respect to g. Denoting, respectively, by g_1 and g_2 the restrictions of g to ξ_1 and ξ_2, from (2.2) we obtain

$$\delta_{J_1} g_1 = 0 \qquad \delta_{J_2} g_2 = 0 \tag{2.3}$$

which may be written as

$$g_1(J_1X, J_1X) = -g_1(X, Y) \qquad g_2(J_2X, J_2Y) = g_2(X, Y) \qquad X, Y \in \mathfrak{X}(\xi). \tag{2.4}$$

Hence (J_1, g_1) is a para-Hermitian structure of ξ_1 and (J_2, g_2) is an indefinite Hermitian structure of ξ_2. Conversely, if ξ_1 and ξ_2 are two supplementary subbundles of ξ such that ξ_1 is endowed with a para-Hermitian structure (J_1, g_1) and ξ_2 with an indefinite Hermitian structure (J_2, g_2), then considering J as given by (2.1) and setting

$$g = g_1 \oplus g_2$$

one obtains a mem-structure on ξ. So we have

Proposition 2.2. *A mem-structure (J, g) on ξ is equivalent to a pair of supplementary subbundles ξ_1 and ξ_2, respectively, endowed with a para-Hermitian structure (J_1, g_1) and an indefinite Hermitian structure (J_2, g_2).*

Remark 2.2. *If (J, g) is a mem-structure on ξ, then we have:* $\operatorname{rank}\xi_1$ *and* $\operatorname{rank}\xi_2$ *are even;* $\operatorname{trace} J_1 = \operatorname{trace} J_2 = 0$*;* $\operatorname{sign} g_1 = 0$.

Setting for a mem-structure (J, g) on ξ:

$$\Omega(X, Y) = g(JX, Y) \qquad \Omega_i(X, Y) = g_i(J_iX, Y) \qquad i = 1, 2$$

it follows that Ω, Ω_1 and Ω_2 are 2-forms, which determine almost-symplectic structures of ξ, ξ_1 and ξ_2, so that

$$\Omega = \Omega_1 \oplus \Omega_2.$$

These 2-forms satisfy

$$\delta_J \Omega = 0 \qquad \delta_{J_1} \Omega_1 = 0 \qquad \delta_{J_2} \Omega_2 = 0. \tag{2.5}$$

Remark 2.3. *The meaning of conditions (2.2), (2.3) and (2.5) is the following: the groups of automorphisms of $\mathfrak{X}(\xi_1)$, $\mathfrak{X}(\xi_2)$ and $\mathfrak{X}(\xi)$ given by*

$$\alpha_t = I_1 \cosh t + J_1 \sinh t \qquad \beta_t = I_2 \cos t + J_2 \sin t \qquad \gamma_t = \alpha_t \oplus \beta_t$$

$t \in \mathbb{R}$, determine actions on the tensor algebras $\mathcal{T}(\xi_1)$, $\mathcal{T}(\xi_2)$ and $\mathcal{T}(\xi)$, which, respectively, preserve the structures (J_1, g_1, Ω_1), (J_2, g_2, Ω_2) and (J, g, Ω).

3. Compatible connections

3.1. The general case

Definition 3.1. *A connection D on the vector bundle ξ is said to be compatible with an em-structure J if*

$$DJ = 0. \tag{3.1}$$

From this it follows that $DP = 0$, hence D preserves the subbundles ξ_1 and ξ_2, i.e. for $X \in \mathfrak{X}(M)$, $Y_1 \in \mathfrak{X}(\xi_1)$, $Y_2 \in \mathfrak{X}(\xi_2)$, one has $D_X Y_1 \in \mathfrak{X}(\xi_1)$, $D_X Y_2 \in \mathfrak{X}(\xi_2)$. Setting then

$$D^1_X Y_1 = D_X Y_1 \qquad D^2_X Y_2 = D_X Y_2$$
$$X \in \mathfrak{X}(M) \qquad Y_1 \in \mathfrak{X}(\xi_1) \qquad Y_2 \in \mathfrak{X}(\xi_2)$$

we have that D^1 and D^2 are, respectively, connections on ξ_1 and ξ_2, so that

$$D_X = D^1_X \circ P_1 + D^2_X \circ P_2 \qquad D^1_X J_1 = 0 \qquad D^2_X J_2 = 0 \qquad X \in \mathfrak{X}(M). \tag{3.2}$$

Conversely, if D^1 and D^2 are, respectively, connections on ξ_1 and ξ_2, then D given as in (3.2) is a connection on ξ satisfying $DP = 0$. If D_1 and D_2 satisfy the respective conditions in (3.2), then D satisfies (3.1) too. Thus, it follows

Proposition 3.1. *A connection D on ξ is compatible with the em-structure J if and only if there exist two connections D^1 on ξ_1 and D^2 on ξ_2, respectively, compatible with the product structure J_1 and the complex structure J_2, so that*

$$D = D^1 \circ P_1 + D^2 \circ P_2. \tag{3.3}$$

Consider now on the subbundles ξ_i of ξ, the operators Φ_{J_i} and Ψ_{J_i} given by

$$(\Phi_{J_i} D^i)_X = \tfrac{1}{2}(D^i_X + J_i^{-1} \circ D^i_X \circ J_i) \qquad (\Psi_{J_i} \mathcal{A}^i)_X = \tfrac{1}{2}(\mathcal{A}^i_X + J_i^{-1} \circ \mathcal{A}^i_X \circ J_i) \tag{3.4}$$

where $X \in \mathfrak{X}(M)$, D^i is a connection on ξ_i, and $\mathcal{A}^i \in \Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$ (*now and in the following we take* $i = 1, 2$). From [1, 13] and proposition 3.1 we obtain

Proposition 3.2. *The set of connections on ξ compatible with the em-structure J is given by*

$$D_X = \{(\Phi_{J_1} D^{\circ 1})_X + (\Psi_{J_1} \mathcal{A}^1)_X\} \circ P_1 + \{(\Phi_{J_2} D^{\circ 2})_X + (\Psi_{J_2} \mathcal{A}^2)_X\} \circ P_2$$

where $X \in \mathfrak{X}(M)$ and $D^{\circ i}$ is an arbitrary fixed connection on ξ_i, $\mathcal{A}^i$ denotes any element of $\Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$, and Φ_{J_i}, Ψ_{J_i} are given by (3.4).

Definition 3.2. *A connection D on ξ is said to be compatible with the mem-structure (J, g) if*

$$DJ = 0 \qquad Dg = 0.$$

From which it follows: $DP = 0$; $D = D^1 \circ P_1 + D^2 \circ P_2$, where D^i are the restrictions of D to ξ_1 and ξ_2; $D^i J_i = 0$; and $D^i g_i = 0$. Conversely, if D^1 and D^2 are connections on ξ_1 and ξ_2, compatible with the para-Hermitian structure (J_1, g_1) and the indefinite Hermitian structure (J_2, g_2), respectively, then the connection D given by (3.3) is compatible with the mem-structure (J, g) on ξ. So, we have

Proposition 3.3. *A connection D on ξ is compatible with the mem-structure (J, g) on ξ, if and only if there are two connections D^1 and D^2 on the subbundles ξ_1 and ξ_2, respectively, compatible with the para-Hermitian structure (J_1, g_1) and the indefinite Hermitian structure (J_2, g_2), so that D is given by (3.3).*

Then setting

$$(\Phi_{g_i} D^i)_X = \tfrac{1}{2}(D^i_X + g_i^{-1} \circ D^i_X \circ g_i) \qquad (\Psi_{g_i} \mathcal{A}^i)_X = \tfrac{1}{2}(\mathcal{A}^i_X + g_i^{-1} \circ \mathcal{A}^i_X \circ g_i) \tag{3.5}$$

we obtain from [1], proposition 3.3 and (2.4)

Proposition 3.4. *The set of connections on ξ compatible with the mem-structure (J, g) is given by*

$$D_X = \left\{((\Phi_{g_1} \circ \Phi_{J_1})D^{\circ 1})_X + ((\Psi_{g_1} \circ \Psi_{J_1})\mathcal{A}^1)_X\right\} \circ P_1$$
$$+\left\{((\Phi_{g_2} \circ \Phi_{J_2})D^{\circ 2})_X + ((\Psi_{g_2} \circ \Psi_{J_2})\mathcal{A}^2)_X\right\} \circ P_2$$

where $D^{\circ i}$ is an arbitrary fixed connection on ξ_i, $\mathcal{A}^i \in \Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$, and Φ_{J_i}, Φ_{g_i}, Ψ_{J_i}, Ψ_{g_i} are given by (3.4) and (3.5).

3.2. The case of the tangent bundle

We now consider the case of ξ being the tangent bundle of the manifold M, i.e. $\xi = (TM, \pi, M)$. In this case, for a mem-structure (J, g) on M, the pair (P, g) is a pseudo-Riemannian almost-product structure on M, and (J_1, g_1), (J_2, g_2), are, respectively, a para-Hermitian [4] and an indefinite Hermitian structure [10] on ξ_1 and ξ_2. If ∇ is a linear connection on M, compatible with P, i.e. $\nabla P = 0$, then its restrictions ∇^1 and ∇^2 to ξ_1 and ξ_2 are connections on these subbundles. If T is the torsion tensor of ∇, we shall call the *torsion tensor* of ∇^i to the tensor fields T^i given by $T^i = P_i \circ T|_{\xi_i}$, or in more detail

$$T^i(X_i, Y_i) = \nabla_{X_i} Y_i - \nabla_{Y_i} X_i - P_i[X_i, Y_i] \qquad X_i, Y_i \in \mathfrak{X}(\xi_i).$$

We call *tensors of nonholonomy* of the distributions ξ_1 and ξ_2 to the tensor fields $S^1 = P_2 \circ T|_{\xi_1}$ and $S^2 = P_1 \circ T|_{\xi_2}$, respectively. We obtain

$$S^1(X_1, Y_1) = -P_2[X_1, Y_1] \qquad S^2(X_2, Y_2) = -P_1[X_2, Y_2].$$

It follows

Proposition 3.5. *The distribution ξ_1 (respectively, ξ_2) is involutive if and only if $S^1 = 0$ (respectively, $S^2 = 0$).*

After some computations we obtain from [3, 10, 14].

Proposition 3.6. *For a mem-structure (J, g) on a manifold M, there exists a unique linear connection ∇ with torsion tensor T, satisfying the conditions*

$$\nabla P = 0 \qquad T(PX, Y) = T(X, PY) \tag{3.6}$$

$$\nabla^i_{X_i} J_i = 0 \qquad \nabla^i_{X_i} g_i = 0 \qquad T^i(J_i X, I_i Y) = T^i(I_i X, J_i Y). \tag{3.7}$$

Definition 3.3. *We shall call the canonical connection associated with the mem-structure (J, g) on the manifold M to the connection given by the conditions (3.6) and (3.7).*

Remark 3.1. *Notice that this connection differs slightly from that given in theorem 5.3 in [14].*

For the canonical connection we obtain from (3.6):

$$\nabla^1_{X_2} Y_1 = P_1[X_2, Y_1] \qquad \nabla^2_{X_1} Y_2 = P_2[X_1, Y_2].$$

Denoting by ξ_1^1, ξ_1^2 the eigen-subbundles of J_1 corresponding to $\varepsilon = +1$, $\varepsilon = -1$, by π_1^1, π_1^2 the projection maps of ξ_1 on ξ_1^1, and ξ_1^2 and by X_1^i, Y_1^i any elements of $\mathfrak{X}(\xi_1^i)$, we obtain from the first equation in (3.7),

$$\nabla^1_{X_1^2} Y_1^1 = \pi_1^1 P_1[X_1^2, Y_1^1] \qquad \nabla^1_{X_1^1} Y_1^2 = \pi_1^2 P_1[X_1^1, Y_1^2]$$
$$g_1(\nabla^1_{X_1^1} Y_1^1, Z_1^2) = X_1^1 g_1(Y_1^1, Z_1^2) - g_1([X_1^1, Z_1^2], Y_1^1)$$
$$g_1(\nabla^1_{X_1^2} Y_1^2, Z_1^1) = X_1^2 g_1(Y_1^2, Z_1^1) - g_1([X_1^2, Z_1^1], Y_1^2).$$

From the second equation in (3.7) above it results, exactly as in [14] (theorem 5.1), the expression for $\nabla^2_{X_2} Y_2$.

For J and g we obtain

$$(\nabla_{X_1} J)Y_1 = 0 \qquad (\nabla_{X_2} J)Y_2 = 0 \qquad (\nabla_{X_1} J)Y_2 = (\nabla^2_{X_1} J_2)Y_2$$
$$(\nabla_{X_2} J)Y_1 = (\nabla^1_{X_1} J_1)Y_1 \qquad (\nabla_{X_1} g)(Y_1, Z_1) = 0 \qquad (\nabla_{X_2} g)(Y_2, Z_2) = 0$$
$$(\nabla_{X_2} g)(Y_1, Z_1) = (L_{X_2} g)(Y_1, Z_1) \qquad (\nabla_{X_1} g)(Y_2, Z_2) = (L_{X_1} g)(Y_2, Z_2)$$

where L stands for the Lie derivative.

4. Structures of electromagnetic type on the total space of a vector bundle

Let $\xi = (E, \pi, M)$ be a vector bundle and (x^j), (y^a), (x^j, y^a), local coordinates in adapted charts on M, ξ and E, respectively. We denote by (∂_j), (e_a), (∂_j, ∂_a) the corresponding local bases, where $\partial_j = \partial/\partial x^j$, $\partial_a = \partial/\partial y^a$, $j = 1, 2, \ldots, m$, $a, b, c = 1, 2, \ldots, n$ (see [2]). Setting for each $z = (x, y) \in E$, $V_z E = \operatorname{Ker} \pi_{*z}$, we obtain the *vertical distribution* and thus the *vertical subbundle* of TE, denoted by VE. Let $C^{\infty v} = \{f^v = f \circ \pi : f \in C^\infty(M)\}$ be the subring of $C^\infty(E)$ naturally isomorphic to $C^\infty(M)$. Setting for each $\mu \in \Lambda^1(\xi)$, locally given by $\mu(x) = \mu_a(z)\, e^a$,

$$\gamma(\mu)(z) = \mu_a(x)\, y^a$$

we obtain a class of functions on E enjoying the property that every vector field $A \in \mathfrak{X}(E)$ is uniquely determined by its values on those functions. The mapping γ may be extended to tensor fields $S \in \mathcal{T}^1_1(\xi)$ by

$$(\gamma S)(\gamma(\mu)) = \gamma(\mu \circ S) \qquad \mu \in \Lambda^1(\xi).$$

If $S(x) = S^a_b(x)\, e_a \otimes e^b$, then $\gamma S(z) = S^a_b(x)\, y^b \partial_a$, i.e. γS is a vertical vector field on E. Now, let D be a connection on ξ and $X \in \mathfrak{X}(M)$, $u \in \mathfrak{X}(\xi)$. Setting

$$X^h(\gamma\mu) = \gamma(D_X \mu) \qquad u^v(\gamma\mu) = \mu(u) \circ \pi \qquad \mu \in \Lambda^1(\xi)$$

we obtain two vector fields X^h and u^v on E, respectively, called the *horizontal lift* of X and the *vertical lift* of u. We have the useful formulae [2]

$$(fX)^h = f^v X^h \qquad (fu)^v = f^v u^v$$
$$[X^h, Y^h] = [X, Y]^h - \gamma R^D_{XY} \qquad [u^v, w^v] = 0[X^h, u^v] = (D_X u)^v$$
$$f \in \mathbb{C}^\infty(M) \qquad X, Y \in \mathfrak{X}(M) \qquad u, w \in \mathfrak{X}(\xi).$$

Now, putting

$$Q(X^h) = X^h \qquad Q(u^v) = -X^v \qquad X \in \mathfrak{X}(M) \qquad u \in \mathfrak{X}(\xi)$$

we obtain an almost-product Q structure on E whose +1 and -1 eigendistributions, are, respectively, called the *horizontal distribution* HE of the connection D and the *vertical distribution* VE of the bundle.

For $f \in \mathcal{T}^1_1(M)$, $\varphi \in \mathcal{T}^1_1(\xi)$, $g \in \mathcal{T}_2(M)$, $\psi \in \mathcal{T}_2(\xi)$, we define the *horizontal lift* or the *vertical lift* f^h, φ^v, g^h, ψ^v, respectively, by

$$\begin{aligned}
&f^h(X^h) = f(X)^h \qquad f^h(u^v) = 0 \\
&\varphi^v(X^h) = 0 \qquad \varphi^v(u^v) = \varphi(u)^v \\
&g^h(X^h, Y^h) = g(X, Y)^v \qquad g^h(X^h, u^v) = g^h(u^v, X^h) = g^h(u^v, w^v) = 0 \\
&\psi^v(X^h, Y^h) = \psi^v(X^h, u^v) = \psi^v(u^v, Y^h) = 0 \qquad \psi^v(u^v, w^v) = \psi(u, w)^v \\
&X, Y \in \mathfrak{X}(M) \qquad u, w \in \mathfrak{X}(\xi).
\end{aligned} \tag{4.1}$$

We then define the *diagonal lifts* J and G for the pairs (f, φ) and (g, ψ) by

$$J = f^h + \varphi^v \qquad G = g^h + \psi^v. \tag{4.2}$$

From (4.1) and (4.2) we have

$$J^n(X^h) = (f^n(X))^h \qquad J^n(u^v) = (\varphi^n(u))^v \qquad n \in \mathbb{N}^*.$$

So $J^4 = I$, that is J is an em-structure on E, if and only if $f^4 = I_1$ and $\varphi^4 = I_2$, that is, either f and φ are both em-structures or one is an em-structure and the other is an almost-product or almost-complex structure, or finally f is an almost-product (respectively, almost-complex) and φ is a complex (respectively, product) structure on M and ξ, respectively. *In the following we only consider the last case*.

Hence, let J be an em-structure on the total space E of ξ given by the diagonal lift in the first equation in (4.2) of an almost-product (respectively, almost-complex) structure f on the base manifold M and a complex (respectively, product) structure φ on the bundle ξ, that is, which satisfy

$$f^2 = \varepsilon I_1 \qquad \varphi^2 = -\varepsilon I_2 \qquad \varepsilon = 1 \text{ (respectively, } \varepsilon = -1)$$

with respect to a connection D on ξ. For the almost-product structure P associated to J, we obtain $P = \varepsilon Q$, that is, P coincides up to the sign with the almost-product structure Q above associated to D.

Now, let G be the diagonal lift in the second equation in (4.2), with respect to D, for the pair (g, ψ) of metrics on M and ξ. From (4.2) we obtain

$$\delta_J G = (\delta_f g)^h + (\delta_\varphi \psi)^v$$

and so $\delta_J G = 0$ if and only if $\delta_f g = 0$ and $\delta_\varphi \psi = 0$. It follows

Proposition 4.1. *The pair (J, G) of diagonal lifts, with respect to a connection D on ξ, of an almost-product (respectively, almost-complex) structure f on M and a complex (respectively, product) structure φ of ξ, and the nondegenerate metrics g on M and ψ on ξ, is a mem-structure on the total space E of ξ if and only if the pair (f, g) is an almost-para-Hermitian (respectively, indefinite almost-Hermitian) structure on M. The pair (φ, ψ) is an indefinite Hermitian (respectively, para-Hermitian) structure on ξ.*

Denoting by ω and τ the 2-forms associated to the structures (f, g) on M and (φ, ψ) on ξ, and by $\Omega_1, \Omega_2, \Omega$, the 2-forms associated to the structures (f^h, g^h) on HE, (φ^v, ψ^v) on VE and (J, G) on TE, we obtain

$$\Omega_1 = \omega^h \qquad \Omega_2 = \tau^v \qquad \Omega = \omega^h \oplus \tau^v.$$

From the hypotheses of proposition 4.1 it follows

$$\delta_f g = 0 \qquad \delta_f \omega = 0 \qquad \delta_\varphi \psi = 0 \qquad \delta_\varphi \tau = 0 \qquad \delta_J G = 0 \qquad \delta_J \Omega = 0.$$

Remark 4.1. *The groups of automorphisms of $\mathfrak{X}(M), \mathfrak{X}(\xi), \mathfrak{X}(E)$, given, respectively, for $\varepsilon = 1$ and $\varepsilon = -1$, by*

$$\alpha_t = I_1 \cosh t + f \sinh t \qquad \beta_t = I_2 \cos t + \varphi \sin t \qquad \gamma_t = \alpha_t^h \oplus \beta_t^h \qquad t \in \mathbb{R}$$
$$\alpha_t = I_1 \cos t + f \sin t \qquad \beta_t = I_2 \cosh t + \varphi \sinh t \qquad \gamma_t = \alpha_t^h \oplus \beta_t^h \qquad t \in \mathbb{R}$$

determine on the tensor algebras $\mathcal{T}(M), \mathcal{T}(\xi)$ and $\mathcal{T}(E)$, actions which preserve the structures $(f, g, \omega), (\varphi, \psi, \tau)$ and (J, G, Ω).

For two connections ∇ on M and D on ξ, we define the *horizontal lift* ∇^h on the subbundle HE and the *vertical lift* D^v on the subbundle VE (each one with respect to the connection D), respectively, by

$$\nabla^h_{X^h} Y^h = (\nabla_X Y)^h \qquad \nabla^h_{u^v} Y^h = 0 \qquad D^v_{X^h} w^v = (D_X w)^v \qquad D^v_{u^v} w^v = 0.$$

Putting them as

$$\mathcal{D}_A X = \nabla^h_A HX + D^v_A VX \qquad A, X \in \mathfrak{X}(E)$$

where H and V denote the horizontal and vertical projectors of TE on HE and VE, we obtain a linear connection $\mathcal{D}$ on E, called the *diagonal lift* of the pair (∇, D) with respect to the connection D (see [2]), whose restrictions to the subbundles $\xi_1 = HE$ and $\xi_2 = VE$ are $\mathcal{D}_1 = \nabla^h$ and $\mathcal{D}_2 = D^v$. The nonvanishing components of the torsion and curvature tensors of $\mathcal{D}$ are given by

$$\begin{aligned} &\mathcal{T}(X^h, Y^h) = T^\nabla(X, Y)^h + \gamma R^D_{XY} \\ &\mathcal{R}_{X^h Y^h} Z^h = (R^\nabla_{XY} Z)^h \qquad \mathcal{R}_{X^h Y^h} u^v = (R^D_{XY} u)^v \end{aligned} \tag{4.3}$$

where T^∇, R^∇ and R^D stand for the torsion tensor of ∇ and the curvature tensors of ∇ and D.

For the covariant derivatives, with respect to $\mathcal{D}$, of the horizontal lift of f and g, and the vertical lift of φ and ψ we obtain

$$\begin{aligned} &\mathcal{D}_{X^h} f^h = (\nabla_X f)^h \qquad \mathcal{D}_{u^v} f^h = 0 \qquad \mathcal{D}_{X^h} g^h = (\nabla_X g)^h \qquad \mathcal{D}_{u^v} g^h = 0 \\ &\mathcal{D}_{X^h} \varphi^v = (D_X \varphi)^v \qquad \mathcal{D}_{u^v} \varphi^v = 0 \qquad \mathcal{D}_{X^h} \psi^v = (D_X \psi)^v \qquad \mathcal{D}_{u^v} \psi^v = 0. \end{aligned}$$

So, for the diagonal lifts J and G of the pairs (f, φ) and (g, ψ), it follows

$$\begin{aligned} &\mathcal{D}_{X^h} J = (\nabla_X f)^h + (D_X \varphi)^v \qquad \mathcal{D}_{u^v} J = 0 \\ &\mathcal{D}_{X^h} G = (\nabla_X g)^h + (D_X \psi)^v \qquad \mathcal{D}_{u^v} G = 0. \end{aligned} \tag{4.4}$$

Hence, $\mathcal{D}J = 0$ if and only if $\nabla f = 0$, $D\varphi = 0$; and $\mathcal{D}G = 0$ if and only if $\nabla g = 0$, $D\psi = 0$. From (4.3) and (4.4) it follows, for $P = J^2$, that $\mathcal{D}P = 0$ and $\mathcal{T} \circ P \times I = \mathcal{T} \circ I \times P$ for any connections ∇ on M and D on ξ. After that we have

$$\begin{aligned} &\nabla^h_{X^h} g^h = (\nabla_X g)^h \qquad D^v_{u^v} \varphi^v = 0 \qquad D^v_{u^v} \psi^v = 0 \\ &\nabla^h_{X^h} f^h = (\nabla_X f)^h \qquad \mathcal{T}^1(f^h X, I_1 Y) = (T^\nabla(fX, I_1 Y))^h \qquad \mathcal{T}^2(\varphi^v X, I_2 Y) = 0 \end{aligned}$$

where $\mathcal{T}^1 = H \circ \mathcal{T}|_{HE}$ and $\mathcal{T}^2 = V \circ \mathcal{T}|_{VE}$. So we obtain

Proposition 4.2. *The diagonal lift $\mathcal{D}$ on E, for the connections ∇ on M and D on ξ, is the canonical connection associated to the mem-structure (J, G) if and only if*

$$\nabla f = 0 \qquad \nabla g = 0 \qquad T^\nabla(fX, Y) = T^\nabla(X, fY)$$

i.e. the connection ∇ is the canonical connection [2, 10] *associated to the almost-para-Hermitian (respectively, indefinite almost-Hermitian) structure (f, g) on M.*

Also from (4.3) and (4.4) we obtain $\mathcal{D}G = 0$ and $\mathcal{T} = 0$ if and only if $\nabla g = 0$, $T^\nabla = 0$, $R^D = 0$ and $D\psi = 0$. Hence we have

Proposition 4.3. *The diagonal lift $\mathcal{D}$ of the pair of connections (∇, D) coincides with the Levi-Civita connection of G if and only if ∇ is the Levi-Civita connection of g, D has vanishing curvature and ψ is covariant constant.*

For the Nijenhuis tensor of J,

$$N_J(A,B) = [JA, JB] + J^2[A,B] - J[JA,B] - J[A,JB] \qquad A, B \in \mathfrak{X}(E)$$

we obtain

$$N_J(X^h, Y^h) = N_f(X,Y)^h + \gamma\big(\varepsilon R^D_{XY} - R^D_{fXfY} + \varphi\circ(R^D_{fXY} + R^D_{XfY})\big)$$
$$N_J(X^h, u^v) = \big(D_{fX}\varphi u - \varepsilon D_X u - \varphi\circ(D_{fX}u + D_X\varphi u)\big)^v \qquad N_J(u^v, w^v) = 0. \tag{4.5}$$

It follows

Proposition 4.4. *The mem-structure J is integrable (i.e. $N_J = 0$, see* [8]*) if and only if f is a product (respectively, a complex) structure in M, the connection D has vanishing curvature and the complex (respectively, product) structure φ on ξ is covariant constant.*

For the exterior differential of the 2-form Ω associated to the mem-structure (J, G) we obtain

$$\mathrm{d}\Omega(X^h, Y^h, Z^h) = \mathrm{d}\omega(X,Y,Z)^v \qquad 3\mathrm{d}\Omega(X^h, Y^h, w^v) = -\gamma(i_w\tau\circ R^D_{XY})$$
$$3\mathrm{d}\Omega(X^h, u^v, w^v) = D_X\tau(u,w)^v \qquad \mathrm{d}\Omega(u^v, v^v, w^v) = 0.$$

Hence

Proposition 4.5. *The almost-symplectic structure Ω associated to the mem-structure (J, G) on E is integrable (i.e.* $\mathrm{d}\Omega = 0$*) if and only if the structure (f, g) is almost-para-Kähler (respectively, indefinite almost Kähler), the connection D has vanishing curvature, and the* 2*-form τ on ξ is covariant constant.*

Finally, we obtain

Proposition 4.6. *For the mem-structure (J, G) on E, the structures J and Ω are simultaneously integrable if and only if the structure (f, g) is a para-Kähler (respectively, indefinite Kähler) structure on M, D has vanishing curvature and the pair (φ, ψ) is covariant constant.*

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